ISRAEL JOURNAL OF MATHEMATICS, Vol. 55, No. 1, 1986

SINGULAR POINTS ON COMPACTIFICATIONS OF A RIEMANN SURFACE

BY

MOSES GLASNER[a] AND MITSURU NAKAI[b,†]

[a]*Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA; and*
[b]*Department of Mathematics, Nagoya Institute of Technology, Gokiso, Showa, Nagoya 466, Japan*

ABSTRACT

A notion of capacity on general compactifications of a Riemann surface is introduced. It is used to establish the connection between singular points on the Royden compactification and singular points on the Kuramochi compactification.

The notion of a potential theory and capacity is well known for the Kuramochi compactification (cf. [1]). In [3] (also cf. [4]) Kuramochi established the existence of a Riemann surface with countably many singular points having zero harmonic measure. In [2] it was shown that it is possible that every Royden harmonic boundary point of a Riemann surface has positive capacity, yet not every point has positive harmonic measure.

The purpose of the present paper is to unify these results. We introduce a notion of capacity for general compactifications, which we call *Dirichlet capacity*. In the case where the compactification is the Kuramochi compactification, Dirichlet capacity coincides with a constant multiple of Kuramochi's capacity and in the case where the compactification is the Royden compactification, Dirichlet capacity of a point coincides with the definition of the capacity of a point given in [2].

The notion of Dirichlet capacity allows us to establish the connection between singular points, i.e., points with positive capacity, on the Kuramochi and Royden boundaries. Roughly speaking, the connection is that a point q in the Kuramochi boundary is singular if and only if there is a singular point in the Royden

†The second named author was supported by Grant-in-Aid for Scientific Research, Japanese Ministry of Education, Science and Culture.

Received December 1, 1985

boundary which goes to q under the natural projection. Combining this with the examples cited above gives added information about both the Kuramochi and Royden boundaries. From Kuramochi's example we see that also the Royden boundary may contain uncountably many singular points and from the example given in [2] we see that the set of minimal points in the Kuramochi boundary may consist entirely of singular points.

The main tool for establishing this result is an intrinsic characterization of the existence of singular points. Let W be a fixed regular boundary neighborhood of a hyperbolic Riemann surface R. For a nonzero continuous function u_0 on R with $u_0 | R - W = 0$, $u_0 | W \in \mathrm{HBD}(W; \partial W)$ consider the linear functional

$$f \to \frac{1}{D(u_0)} D(f, u_0) \tag{A.1}$$

on $M(W; \partial W)$, the functions in the Royden algebra of R vanishing on $R - W$. The functional in (A.1) is multiplicative, i.e.,

$$\frac{1}{D(u_0)} D(f_1 f_2, u_0) = \frac{1}{D(u_0)} D(f_1, u_0) \frac{1}{D(u_0)} D(f_2, u_0), \tag{A.2}$$

if and only if u_0 is the capacitary potential of a singular point. The fact that this is true both in the Royden and Kuramochi settings and the fact that there is a one-to-one correspondence between singular points under the natural projection lead to the main result.

Notations and definitions

1. We shall follow the terminology concerning potential theory on the Kuramochi compactification given in [1] and that on the Royden compactification given in [5] rather closely. Since we will frequently deal with these two compactifications simultaneously, some additional notations are needed.

We consider R a hyperbolic Riemann surface and W a fixed regular connected boundary neighborhood, i.e. W is connected; $R - W$ is compact and ∂W consists of piecewise analytic curves. We denote by R^* a general compactification and by $R^*_{\mathscr{R}}$ ($R^*_{\mathscr{K}}$) the Royden (Kuramochi, respectively) compactification of R. For a set $A \subset R$ we use the symbols $\mathrm{cl}(A)$ and ∂A for its closure and boundary with respect to R. We let $W^* = R^* - (R - W)$. For a set $A \subset R^*$ we use $\bar{A}$ to denote its closure with respect to R^*. We reserve the symbol Δ for the Royden harmonic boundary.

The space of Dirichlet functions on R is denoted by $D(R)$ and the Royden algebra by $M(R)$. In addition we use $D(W; \partial W)$ ($M(W; \partial W)$) to denote the functions in $D(R)$ ($M(R)$, respectively) which vanish on $R - W$. For functions $u, v \in D(R)$ the Dirichlet inner product is denoted by $D(u,v)$ and the Dirichlet (semi-)norm is $D(u)^{1/2} = D(u,u)^{1/2}$. The subspace of Dirichlet potentials $D_0(R)$ of $D(R)$ is known to be the closure in Dirichlet norm of the C^∞ functions with compact support. The potential subalgebra $M_\Delta(R)$ of $M(R)$ is the closure in Dirichlet norm and uniform convergence on compact subsets of the C^∞ functions with compact support. Also set $M_\Delta(W;\partial W) = M_\Delta(R) \cap M(W;\partial W)$. It is known that the functions in $M(R)$ which vanish identically on Δ are precisely the functions in $M_\Delta(R)$. This leads immediately to the following generalized Stokes' formula:

$$D(u,f) = 0 \tag{1}$$

for any $u \in \mathrm{HBD}(W;\partial W)$ and $f \in M_\Delta(W;\partial W)$.

2. We will deal with several notions of capacity in this paper: $C(\cdot\,; W)$, the usual notion of capacity defined on subsets of W, $\tilde{C}(\cdot)$, the Kuramochi capacity defined on subsets of $R^*_{\mathscr{K}}$, and $c(\cdot)$, the Dirichlet capacity defined on subsets of a general compactification R^* of R. We begin by establishing some results related to the capacity $C(\cdot\,; W)$.

LEMMA. *Let $\mathcal{O}$ be a relatively compact subset of W and consider*

$$F = \{\phi \in D_0(W); \phi \geqq 1 \text{ q.e. on } \mathrm{cl}(\mathcal{O})\}.$$

Then

$$2\pi C(\mathcal{O}; W) = \inf_{\phi \in F} D(\phi).$$

Here and throughout we use the abbreviation q.e. for quasi-everywhere.

For the proof we choose a minimizing sequence $\{\phi_n\} \subset F$. Clearly this sequence is D-Cauchy. Then by [1, Hilfssatz 7.8] a subsequence, again denoted by $\{\phi_n\}$, converges in Dirichlet norm and q.e. on W to a function $\omega_0 \in D_0(W)$. Moreover, we see that

$$D(\omega_0, f) = 0, \tag{2}$$

for every $f \in D_0(W)$ with $f = 0$ q.e. on $\mathrm{cl}(\mathcal{O})$. By [1, Hilfssatz 5.5] the capacitary potential for $\mathcal{O}$ is given by $p^{\kappa^{\mathcal{O}}}$. Then by [1, Satz 7.2] we have $D(p^{\kappa^{\mathcal{O}}}) = 2\pi C(\mathcal{O})$. Since $p^{\kappa^{\mathcal{O}}} \in F$ and actually $p^{\kappa^{\mathcal{O}}} = 1$ q.e. on $\mathrm{cl}(\mathcal{O})$, we see that $D(p^{\kappa^{\mathcal{O}}}) = 2\pi C(\mathcal{O}) \geqq D(\omega_0)$.

For the reverse inequality we use the fact that

$$D(p^{\kappa'}, f) = 2\pi \int f d\kappa'$$

for every $f \in D_0(W)$ (cf. [1, p. 79]). In particular, $D(p^{\kappa'}, p^{\kappa'} - \omega_0) \leqq 0$. Thus, $D(p^{\kappa'}) \leqq D(\omega_0)$. We have shown that $p^{\kappa'} = \omega_0$.

3. Note that the capacitary potential ω_0 for $\mathcal{O}$ relative to W has a continuous extension to $\mathrm{cl}(W) - \partial\mathcal{O}$ and is harmonic on $W - \mathrm{cl}(\mathcal{O})$. Also, in case $\partial\mathcal{O}$ is piecewise analytic ω_0 is continuous on $\partial\mathcal{O}$ and thus may be viewed as being in $M(W; \partial W)$. In the general case we will need at least to be able to approximate ω_0 by continuous capacitary potentials. For this purpose we consider an exhaustion $\{R_n\}$ of $R - \mathrm{cl}(\mathcal{O})$ such that $R - W \subset R_0$. Let $W_n = R_n \cap W$ and $\mathcal{O}_n = W - \mathrm{cl}(W_n)$. Then $\mathcal{O}_n$ is an open set in W containing $\mathrm{cl}(\mathcal{O})$ and $\partial\mathcal{O}_n$ is piecewise analytic. Let ω_n be the capacitary potential of $\mathcal{O}_n$ relative to W.

LEMMA. *Given $\delta > 0$ and a compact set $K \subset \mathrm{cl}(W) - \mathrm{cl}(\mathcal{O})$, there is a nonnegative function $v \in D_0(W) \cap M(W; \partial W)$ which is harmonic on K, $v \mid \mathcal{O} = 1$ and $D(v) < 2\pi C(\mathcal{O}; W) + \delta$.*

Consider the sequence $\{\omega_n\}$ of capacitary potentials for $\{\mathcal{O}_n\}$ relative to W. Note that for each pair of positive integers m, n we have $\omega_{m+n} - \omega_n \mid \mathrm{cl}(\mathcal{O}_{m+n}) = 0$. Thus by (2) we see that $D(\omega_{m+n}, \omega_{m+n} - \omega_n) = 0$ and consequently $0 \leqq D(\omega_n) - D(\omega_{m+n})$. This shows that $\{D(\omega_n)\}$ is decreasing and that $\{\omega_n\}$ is D-Cauchy. Thus by [1, Hilfssatz 7.8] a subsequence, again denoted by $\{\omega_n\}$, converges to $\hat{\omega}_0 \in D_0(W)$ in D-norm and q.e. on W. Since $\omega_n \geqq \omega_{m+n}$, the Harnack principle implies that ω_0 is harmonic on $W - \mathrm{cl}(\mathcal{O})$. Since $\hat{\omega}_0 - \omega_0 \in D_0(W)$, it is bounded by a potential on W. Thus, in view of $\hat{\omega}_0 - \omega_0$ being harmonic on $W - \mathrm{cl}(\mathcal{O})$ and having boundary value 0 at every regular boundary point of $\partial\mathcal{O}$, the maximum principle implies that it is identically 0. We conclude that $\{\omega_n\}$ converges to ω_0 in Dirichlet norm and q.e. on W. In particular, $D(\omega_n) \downarrow D(\omega_0)$. Thus the desired function v can be chosen to be ω_n with n sufficiently large.

4. The following will be essential for later purposes.

LEMMA. *Let $\mathcal{O}$ be an open subset of W with $\mathrm{cl}(\mathcal{O}) \cap \partial W = \varnothing$. Given an $\varepsilon > 0$ and $f \in D(W; \partial W)$ with $|f| \leqq 1$, $f \mid \mathcal{O} = 0$ and f continuous on $R - \partial\mathcal{O}$, there is a $g \in M(W; \partial W)$ with $g \mid \mathcal{O} = 0$ and $D(g - f) < \varepsilon$.*

Since $f \mid \mathcal{O} = 0$, we may choose a regular region W_0 such that $D_{W - W_0}(f) < \varepsilon/4$,

$W_0 \subset W - \mathcal{O}$ and $W - \mathrm{cl}(W_0)$ is connected. Let $\{R_k\}$ be a regular exhaustion of R such that $W_0 \cup (R - W) \subset R_0$. Set $W_k = R_k \cap W$. We temporarily view f as being a quasi-continuous function on the Riemann surface $V = W - \mathrm{cl}(W_0)$. For each $k \geqq 1$ there is an open subset G_k of V such that f is continuous on $V - G_k$ and $C(G_k; V) < \varepsilon/\pi 2^{k+4}$. Set $H_k = G_k \cap (W_{k+3} - \mathrm{cl}(W_{k+1}))$. Then also $C(H_k; V) < \varepsilon/\pi 2^{k+4}$. By Lemma 3, there is a nonnegative function $v_k \in D_0(V) \cap M(V; \partial V)$ with v_k harmonic on $\mathrm{cl}(W_k \cap V)$, $v_k \mid H_k = 1$, and $D(v_k) < \varepsilon/2^{k+2}$.

We now consider the sequence $\phi_n = \sum_1^n v_k$. Also set $\phi = \sum_1^\infty v_k$. According to [1, Hilfssatz 7.8] we have $\phi \in D_0(V)$ with $D(\phi) < \varepsilon/4$. Moreover, the function $\phi_n - \phi_m$ is harmonic on $\mathrm{cl}(W_m \cap V)$ for any positive integer m and every $n > m$. We see that ϕ is continuous on $\mathrm{cl}(V)$ and $\phi \mid \partial V = 0$. Since $v_k \mid H_k = 1$, $\phi \mid \cup H_k \geqq 1$. Replacing ϕ by $\phi \cap 1$ gives a function in $M(V; \partial V)$ with $D(\phi) < \varepsilon/4$ and $\phi \mid \cup H_k = 1$. Recall that f is continuous on $R - \cup H_k$. Thus the desired function g is $f - \phi f$. Indeed,

$$D(f-g) = D(\phi f) = D_V(\phi f) \leqq 2D_V(\phi) + 2D_V(f) < \varepsilon.$$

Dirichlet capacity

5. We proceed to define the Dirichlet capacity of a compact set in W^*. Consider an open set $\mathcal{O} \subset R$ with $\mathrm{cl}(\mathcal{O}) \subset W$ and the family

$$E = \{\phi \in M(W; \partial W); \phi \mid \mathcal{O} \geqq 1\}.$$

We can find a regular region Ω such that $R - W \subset \Omega \subset R - \mathcal{O}$ and consequently a function $e \in M(R)$ such that $e \mid R - W = 0$ and $e \mid R - \Omega = 1$. Thus $E \neq \varnothing$ and we can define

$$d(\mathcal{O}) = \inf_{\phi \in E} D(\phi). \tag{3}$$

This definition gives a real-valued function on the open subsets of W with $\mathrm{cl}(\mathcal{O}) \cap (R - W) = \varnothing$. Clearly, $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $d(\mathcal{O}_1) \leqq d(\mathcal{O}_2)$.

For a compact set K in W^* we set

$$c(K) = \inf\{d(U \cap R); U \text{ open in } W^*, K \subset U, \mathrm{cl}(U \cap R) \subset W\}, \tag{4}$$

which we call the *Dirichlet capacity* of K. In case $R^* = R^*_{\mathfrak{R}}$ ($R^*_{\mathcal{K}}$) we use the notation $c_{\mathfrak{R}}(K)$ ($c_{\mathcal{K}}(K)$, respectively) for $c(K)$. We have the following

LEMMA. *Let K_1, K_2 be compact sets in W^*. Then*

(i) *$K_1 \subset K_2$ implies that $c(K_1) \leqq c(K_2)$,*

(ii) $c(K_1 \cup K_2) + c(K_1 \cap K_2) \leqq c(K_1) + c(K_2)$,

(iii) *given* $\varepsilon > 0$ *there is an open set* U *containing* K_1 *such that for every* K *compact in* W^* *contained in* U

$$c(K) \leqq c(K_1) + \varepsilon.$$

This lemma says that $c(K)$ is a capacity in the sense of Choquet. Property (i) is obvious. For (ii) assume $\delta > 0$ is given. Then we may choose $\mathcal{O}_i$ open in W^* with $\mathrm{cl}(\mathcal{O}_i \cap R) \subset W$, containing K_i and $f_i \in M(W; \partial W)$ with $f_i \mid \mathcal{O}_i \geqq 1$ such that $D(f_i) < c(K_i) + \delta$ $(i = 1,2)$. Set $f = f_1 \cap f_2$ and $g = f_1 \cup f_2$. Then

$$D(f) + D(g) = D(f_1) + D(f_2) < c(K_1) + c(K_2) + 2\delta.$$

Also $f \mid \mathcal{O}_1 \cap \mathcal{O}_2 \geqq 1$ and $g \mid \mathcal{O}_1 \cup \mathcal{O}_2 \geqq 1$. Thus $d(\mathcal{O}_1 \cap \mathcal{O}_2) \leqq D(f)$, and $d(\mathcal{O}_1 \cup \mathcal{O}_2) \leqq D(g)$. We see that $c(K_1 \cup K_2) + c(K_1 \cap K_2) \leqq c(K_1) + c(K_2) + 2\delta$, which establishes the assertion.

For property (iii) simply choose U open with $K_1 \subset U$ and $d(U) \leqq c(K_1) + \varepsilon$. Then by definition $c(K) \leqq d(U)$.

Dirichlet capacitary potentials

6. In order to establish the existence and the properties of the function giving the infimum to the $\{D(\phi); \phi \in E\}$ we need the following auxiliary notions. Consider

$$\hat{E} = \{\phi \in D(W; \partial W); \phi \geqq 1 \text{ q.e. on } \mathrm{cl}(\mathcal{O})\}$$

and

$$\hat{d}(\mathcal{O}) = \inf\{D(\phi); \phi \in \hat{E}\}. \tag{5}$$

Note that $\hat{d}(\mathcal{O})$ is not changed if the infimum is taken over functions satisfying the additional restriction $0 \leqq \phi \leqq 1$. Using [1, Satz 15.1] applied with $F = (R - W) \cup \mathrm{cl}(\mathcal{O})$ we can easily see that there is a function $\hat{u}_0$ $(= e^F$ in the notation of [1], where e is defined at the beginning of section 5) satisfying the following properties:

(a) $\hat{u}_0 \in D(W; \partial W)$ with $\hat{u}_0 = 1$ q.e. on $\mathrm{cl}(\mathcal{O})$,

(b) $D(\hat{u}_0, f) = 0$ for every $f \in D(W; \partial W)$ with $f \mid \mathrm{cl}(\mathcal{O}) = 0$,

(c) $\hat{d}(\mathcal{O}) = D(\hat{u}_0)$,

(d) $\hat{u}_0$ is harmonic on $W - \mathrm{cl}(\mathcal{O})$,

(e) $\hat{u}_0$ is continuous at every point of ∂W and the function $\hat{u}_0 \mid W - \mathrm{cl}(\mathcal{O})$ has the boundary value 1 at every point of $\partial \mathcal{O}$ which is regular for $W - \mathrm{cl}(\mathcal{O})$,

(f) properties (a) and (b) or (a) and (c) alone determine $\hat{u}_0$ quasi-everywhere.

Clearly, $\hat{d}(\mathcal{O}) \leqq d(\mathcal{O})$. Our goal is to show that actually equality holds and to establish an approximation property for $d(\mathcal{O})$.

7. We now exhibit a function u_0 having $D(u_0) = d(\mathcal{O})$. To this end consider a minimizing sequence $\{u_n\} \subset E$, i.e., $\{u_n\} \subset M(W; \partial W)$ with $u_n \mid \mathcal{O} = 1$ and $D(u_n) \to d(\mathcal{O})$. This sequence is easily seen to be D-Cauchy and therefore by [1, Hilfssatz 15.1] applied with $F = R - W$ there is a function $u_0 \in D(W; \partial W)$ such that u_0 is the D-limit of $\{u_n\}$. Thus

$$D(\hat{u}_0) = \hat{d}(\mathcal{O}) \leqq d(\mathcal{O}) = D(u_0). \tag{6}$$

By applying Hilfssatz 15.1 to the sequence $\{1 - u_n\}$ with $F = \mathrm{cl}(\mathcal{O})$ we see that $1 = u_0$ q.e. on $\mathrm{cl}(\mathcal{O})$. We may assume that $u_0 \mid \mathcal{O} = 1$ and that $u_0 \mid R - \mathrm{cl}(W) = 0$.

Since u_0 is the D-limit of any minimizing sequence in $M(W; \partial W)$, we see that u_0 is harmonic on $W - \mathrm{cl}(\mathcal{O})$ and vanishes continuously on ∂W. Indeed, if $\{u_n\}$ is a minimizing sequence, then by replacing u_n by $(u_n \cap 1) \cup 0$ we may assume $0 \leqq u_n \leqq 1$. Let V be any regular region contained in $R - \mathrm{cl}(\mathcal{O})$ with $\partial W \subset V$. Then for each n let u_n^V be the function in $M(W; \partial W)$ which agrees with u_n on $R - (V \cap W)$ and is harmonic in $V \cap W$. By the Dirichlet principle $\{u_n^V\}$ is also a minimizing sequence. Since $\{u_n^V\}$ is uniformly bounded on V and converges in D-norm on V to u_0, we see that a subsequence converges to u_0 uniformly on compact subsets of V, which in particular means that u_0 is harmonic on $V \cap W$ and vanishes continuously on $V \cap \partial W$.

Now choose any $g \in M(W; \partial W)$ with $g \mid \mathcal{O} = 0$. Then for any real number t and positive integer n the function $u_n + tg \in E$. Hence $D(u_0) \leqq D(u_n) + 2tD(u_n, g) + t^2 D(g)$. Letting $n \to \infty$ we obtain that $0 \leqq 2tD(u_0, g) + t^2 D(g)$ for any real number t. Thus

$$D(u_0, g) = 0, \tag{7}$$

for any $g \in M(W; \partial W)$ with $g \mid \mathcal{O} = 0$.

Note that $\hat{u}_0 - u_0$ is bounded, is continuous on $R - \partial\mathcal{O}$, and belongs to $D(W; \partial W)$ with $\hat{u}_0 - u_0 \mid \mathcal{O} = 0$. Thus by Lemma 4 it can be approximated in Dirichlet norm by a sequence $\{g_n\} \subset M(W; \partial W)$ with $g_n \mid \mathcal{O} = 0$. By applying (7) to g_n and taking the limit we have $D(u_0, \hat{u}_0 - u_0) = 0$. This implies that $D(u_0) \leqq D(\hat{u}_0)$. Together with (6) and the uniqueness property of u_0 stated in section 6 we conclude that $u_0 = \hat{u}_0$. We shall call u_0 the *Dirichlet capacitary potential* for $\mathcal{O}$.

We have established the following.

LEMMA. *For an open set $\mathcal{O} \subset R$ the quantities $d(\mathcal{O})$, $\hat{d}(\mathcal{O})$ defined in* (3) *and* (5), *respectively, are identical.*

8. Let $\{\mathcal{O}_n\}$ be an exhaustion of $\mathcal{O}$ by regular open sets and let w_n be the Dirichlet capacitary potential for $\mathcal{O}_n$. Since $\partial\mathcal{O}_n$ is piecewise analytic, $w_n \in M(W;\partial W)$. Moreover, $w_n \mid \mathcal{O}_n = 1$ and $D(w_n, f) = 0$ for every $f \in D(W;\partial W)$ with $f = 0$ q.e. on $\mathrm{cl}(\mathcal{O}_n)$. Since every function $\phi \in E$ is a function competing for $d(\mathcal{O}_n)$, we see that $D(u_0) \geqq D(w_n)$. For positive integers m, n, $w_{m+n} - w_n \mid \mathcal{O}_n = 0$. Thus, $D(w_n, w_{m+n} - w_n) = 0$. We see that $\{D(w_n)\}$ is a bounded increasing sequence and consequently $\{w_n\}$ is D-Cauchy. By [1, Hilfssatz 15.1] there is a function $w_0 \in D(W;\partial W)$ which is the limit of this sequence. Note that for any n, $w_0 \mid \mathcal{O}_n = 1$ and hence $w_0 = 1$ q.e. on $\mathrm{cl}(\mathcal{O})$.

Let $f \in D(W;\partial W)$ with $f = 0$ q.e. on $\mathrm{cl}(\mathcal{O})$. Then by the minimizing property of w_n we have $D(w_n, f) = 0$. Letting $n \to \infty$ gives $D(w_0, f) = 0$. By the uniqueness property of u_0, we see that $w_0 = u_0$. In particular, we have established the

LEMMA. *For an exhaustion $\{\mathcal{O}_n\}$ of $\mathcal{O}$ by regular open sets*

$$\lim d(\mathcal{O}_n) = d(\mathcal{O}). \tag{8}$$

Dirichlet capacity of open sets

9. The *Dirichlet capacity* of an open subset $\mathcal{O} \subset W^*$ is defined as follows:

$$c(\mathcal{O}) = \sup\{c(K);\ K \text{ is compact in } W^* \text{ and } K \subset \mathcal{O}\},$$

where $c(K)$ was defined in (4). In the special case where $\mathcal{O} \subset R$ the definition of Dirichlet capacity takes the form

$$c(\mathcal{O}) = \sup\{c(K);\ K \text{ is compact in } W \text{ and } K \subset \mathcal{O}\}, \tag{9}$$

where $c(K)$ can be expressed more simply as

$$c(K) = \inf\{d(U);\ U \text{ open in } W \text{ and } K \subset U\}. \tag{10}$$

In order to work with the notion of Dirichlet capacity we need to establish the fundamental fact that $c(\mathcal{O}) = c(\mathcal{O} \cap R)$ for $\mathcal{O}$ open in W^*. We prove this in a sequence of lemmas starting with the

LEMMA. *If $\mathcal{O}$ is a regular open set with* $\mathrm{cl}(\mathcal{O}) \subset W$, *then*

$$c(\mathrm{cl}(\mathcal{O})) = d(\mathcal{O}). \tag{11}$$

For the proof we first observe that for any U open in W with $\mathrm{cl}(\mathcal{O}) \subset U$ the inequality $d(U) \geqq d(\mathcal{O})$ holds. In view of (10), taking the infimum over all such U gives $c(\mathrm{cl}(\mathcal{O})) \geqq d(\mathcal{O})$ To establish the reverse inequality let u_0 be the Dirichlet capacitary potential for $\mathcal{O}$. Since $\mathcal{O}$ is regular, u_0 is continuous on R. For each integer $n > 0$, set $a_n = (n-1)/n$. Since u_0 is harmonic on $u_0^{-1}(0,1)$, we may alter a_n slightly so that $u_0^{-1}(a_n)$ contains no critical points of u_0. Set $U_n = u_0^{-1}(a_n, 1)$ and note that

$$u_n = (u_0 \cap a_n)/a_n$$

is the Dirichlet capacitary potential for U_n. Thus, $d(U_n) = D(u_n)$ approaches $D(u_0) = d(\mathcal{O})$ as $n \to \infty$. Since $\mathrm{cl}(\mathcal{O}) \subset U_n$ for every n we conclude by (10) that $c(\mathrm{cl}(\mathcal{O})) \leqq \lim d(U_n) = d(\mathcal{O})$, which completes the proof.

10. We shall now show that open subsets of W are capacitable.

LEMMA. *If $\mathcal{O}$ is an open subset of W with $\mathrm{cl}(\mathcal{O}) \subset W$, then*

$$c(\mathcal{O}) = d(\mathcal{O}). \tag{12}$$

Let $K \subset \mathcal{O}$ be any compact set. Then by (10) we have $c(K) \leqq d(\mathcal{O})$ and by taking the supremum over all such K we obtain $c(\mathcal{O}) \leqq d(\mathcal{O})$. Now let $\{\mathcal{O}_n\}$ be an exhaustion of $\mathcal{O}$ by regular open sets. Then by (9), (11) and Lemma 8 we see that

$$c(\mathcal{O}) \geqq \lim c(\mathrm{cl}(\mathcal{O}_n)) = \lim d(\mathcal{O}_n) = d(\mathcal{O}).$$

By combining Lemmas 9 and 10 we obtain the

COROLLARY. *If $\mathcal{O}$ is a regular open subset of W with $\mathrm{cl}(\mathcal{O}) \subset W$, then*

$$c(\mathrm{cl}(\mathcal{O})) = c(\mathcal{O}). \tag{13}$$

And in view of Lemma 8 we have

COROLLARY. *If $\mathcal{O}$ is an open subset of W with $\mathrm{cl}(\mathcal{O}) \subset W$ and $\{\mathcal{O}_n\}$ is an exhaustion of $\mathcal{O}$ by regular open sets, then*

$$\lim c(\mathcal{O}_n) = c(\mathcal{O}). \tag{14}$$

11. We can now establish the

LEMMA. *If $\mathcal{O}$ is open in W^* and $\mathrm{cl}(\mathcal{O} \cap R) \subset W$, then*

$$c(\mathcal{O}) = c(\mathcal{O} \cap R). \tag{15}$$

For the proof let K be any compact set with $K \subset \mathcal{O}$. Then by (4), $c(K) \leqq d(\mathcal{O} \cap R)$. Taking the supremum over all such K gives $c(\mathcal{O}) \leqq d(\mathcal{O} \cap R)$. By Lemma 10 we conclude that $c(\mathcal{O}) \leqq c(\mathcal{O} \cap R)$. since the reverse inequality is trivial, the proof is complete.

Combining this result with (14) gives the

COROLLARY. *If $\mathcal{O}$ is an open subset of W^* with* $\mathrm{cl}(\mathcal{O} \cap R) \subset W$ *and* $\{\mathcal{O}_n\}$ *is an exhaustion of* $\mathcal{O} \cap R$ *by regular open sets, then*

$$\lim c(\mathcal{O}_n) = c(\mathcal{O}).$$

We arrive at the following result.

PROPOSITION. *If K is a compact set in W^*, then*

$$c(K) = \inf\{c(\mathcal{O});\ \mathcal{O} \text{ is open in } W^*, \mathrm{cl}(\mathcal{O} \cap R) \subset W \text{ and } K \subset \mathcal{O}\}. \tag{16}$$

The proof is now a simple matter of piecing Lemma 10 together with the above. Indeed, by (15) and (12), $c(\mathcal{O}) = d(\mathcal{O} \cap R)$, which in view of (4) gives (16).

Dirichlet capacity on the Royden compactification

12. For a point p on the Royden harmonic boundary Δ of R, the Dirichlet capacity $c_{\mathcal{R}}(p)$ of p is defined in (4). In [2] we defined the capacity of a point $p \in \Delta$ by

$$\mathrm{cap}(p) = \inf\{D(v);\ v \in M(W; \partial W) \text{ with } v(p) \geqq 1\}. \tag{17}$$

We shall establish the

PROPOSITION. *For $p \in \Delta$, $c_{\mathcal{R}}(p) = \mathrm{cap}(p)$.*

By (4) we can choose a sequence $\{U_n\}$ of open sets in W^* with $\mathrm{cl}(U_n \cap R) \subset W$, $p \in U_n$ and $c_{\mathcal{R}}(p) = \lim d(U_n \cap R)$. Furthermore, by (3) we can choose $\{f_n\} \subset M(W; \partial W)$ with $f_n \mid U_n \cap R \geqq 1$ and $D(f_n) \leqq d(U_n \cap R) + n^{-1}$. By the denseness of $U_n \cap R$ in U_n, we also have $f_n(p) \geqq 1$. Thus $\mathrm{cap}(p) \leqq D(f_n)$. Letting $n \to \infty$ gives $\mathrm{cap}(p) \leqq c_{\mathcal{R}}(p)$.

For the proof of the converse, choose $\{u_n\} \subset M(W; \partial W)$ with $u_n(p) \geqq 1$ and $\lim D(u_n) = \mathrm{cap}(p)$. Then set $a_n = (n-1)/n$ and $U_n = \{p \in R^*; u_n(p) > a_n\}$. Also set $f_n = (u_n \cap a_n)/a_n$. Then $D(f_n) \geqq d(U_n \cap R) \geqq c_{\mathcal{R}}(p)$. We conclude that $\mathrm{cap}(p) = \lim D(u_n) = \lim D(f_n) \geqq c_{\mathcal{R}}(p)$.

13. Assume that $p \in \Delta$ with $c_{\mathcal{R}}(p) > 0$. We shall show that there is a function $u_0 \in M(W; \partial W)$, $u_0(p) = 1$ with $D(u_0) = c_{\mathcal{R}}(p)$. To this end let $\{u_n\}$ be a minimizing sequence, i.e., $\{u_n\} \subset M(W; \partial W)$, $u_n(p) \geqq 1$ and $\lim d(u_n) = c_{\mathcal{R}}(p)$. We see that $\{u_n\}$ is D-Cauchy. According to [1, Hilfssatz 15.1] there is a function $u_0 \in D(W; \partial W)$ which is the D-limit of $\{u_n\}$. Then

$$D(u_0, f) = 0, \tag{18}$$

for every $f \in M(W;\partial W)$ with $f(p)=0$. By essentially the same argument as in section 7 we see that $u_0 \in \mathrm{HD}(W;\partial W)$ and $0 \leqq u_0 \leqq 1$.

Now set $\alpha = u_0(p)$. Then $0 \leqq \alpha \leqq 1$. For every n, $\alpha u_n - u_0$ belongs to $M(W;\partial W)$ and vanishes at p. Thus by (18) we have that $D(u_0, \alpha u_n) = D(u_0)$. Letting $n \to \infty$ gives $\alpha D(u_0) = D(u_0)$, which in view of $c_{\mathcal{R}}(p) > 0$, implies that $\alpha = 1$. We summarize the above in the

LEMMA. *If $c_{\mathcal{R}}(p) > 0$, then there is a $u_0 \in \mathrm{HD}(W;\partial W)$ satisfying*
(i) $u_0(p) = 1$,
(ii) $D(u_0) = c_{\mathcal{R}}(p)$,
(iii) $D(u_0, f) = 0$, *for every $f \in M(W;\partial W)$ with $f(p) = 0$.*
Moreover, u_0 is uniquely determined by (i) *and* (ii) *or* (i) *and* (iii).

We call u_0 the *Dirichlet capacitary potential* for p.

Kuramochi capacity and Dirichlet capacity

14. We shall show that Dirichlet capacity on the Kuramochi compactification and the Kuramochi capacity are essentially the same.

PROPOSITION. *For any compact set $K \subset W^*$,*

$$c_{\mathcal{K}}(K) = 2\pi \tilde{C}(K), \tag{19}$$

and for any open set $\mathcal{O}$ with $\mathcal{O} \subset W^*$ and $\mathrm{cl}(\mathcal{O} \cap R) \subset W$

$$c_{\mathcal{K}}(\mathcal{O}) = 2\pi \tilde{C}(\mathcal{O}). \tag{20}$$

In view of Proposition 11 and the corresponding property for the Kuramochi capacity (cf. [1, p. 188]), it suffices to establish (20). We begin the proof with the following

LEMMA. *If $\mathcal{O}$ is a regular open set in W, then*

$$c_{\mathcal{K}}(\mathcal{O}) = 2\pi \tilde{C}(\mathcal{O}).$$

According to the equilibrium principle (cf. [1, Satz 17.6]) there is a canonical measure $\tilde{\kappa}^{\bar{\mathcal{O}}}$ on $\bar{\mathcal{O}}$ such that $\tilde{p}^{\kappa^{\bar{\mathcal{O}}}} \leqq 1$, $\tilde{p}^{\kappa^{\bar{\mathcal{O}}}} = 1$ q.e. on $\bar{\mathcal{O}}$ and

$$\tilde{C}(\mathcal{O}) = \tilde{\kappa}^{\bar{\mathcal{O}}}(K) = \|\tilde{\kappa}^{\bar{\mathcal{O}}}\|^2.$$

Moreover, by [1, Satz 17.3]

$$D(\tilde{p}^{\kappa^{\bar{\mathcal{O}}}}, f) = 2\pi \int f d\kappa^{\bar{\mathcal{O}}}, \tag{21}$$

for every $f \in D(W; \partial W)$ which is continuous on R^*. In particular we see from (21) that $D(\tilde{p}^{\kappa^{\bar{\mathcal{O}}}}) = 2\pi\tilde{C}(\bar{\mathcal{O}})$. Note that the Dirichlet capacitary potential u_0 is in the class N (cf. [1, p. 167]) and hence is continuous on $R^*_{\mathcal{K}}$. Thus, by (21) $D(\tilde{p}^{\kappa^{\bar{\mathcal{O}}}}, u_0 - \tilde{p}^{\kappa^{\bar{\mathcal{O}}}}) = 0$ and consequently, $D(\tilde{p}^{\kappa^{\bar{\mathcal{O}}}}) \leqq D(u_0)$. By the uniqueness property of u_0, we see that $u_0 = \tilde{p}^{\kappa^{\bar{\mathcal{O}}}}$, which establishes the lemma.

15. We return to the proof of (20). We note that according to [1, Folgesatz 17.6]

$$\tilde{C}(\mathcal{O}) = \sup\{\tilde{C}(K); K \subset \mathcal{O} \cap R, K \text{ is compact}\}. \tag{22}$$

Let $\{\mathcal{O}_n\}$ be an exhaustion of $\mathcal{O} \cap R$ by regular open sets. Then by Corollary 11 and Lemma 14 we have

$$c_{\mathcal{K}}(\mathcal{O}) = \lim c_{\mathcal{K}}(\mathcal{O}_n) = \lim 2\pi\tilde{C}(\mathcal{O}_n) \leqq 2\pi\tilde{C}(\mathcal{O}).$$

For the proof of the reverse inequality we choose a sequence of compact sets $\{K_j\}$ of $\mathcal{O} \cap R$ such that $\tilde{C}(\mathcal{O}) = \lim \tilde{C}(K_j)$ and also a sequence of positive integers $\{n_j\}$ with $K_j \subset \mathcal{O}_{n_j}$. Then

$$2\pi\tilde{C}(K_j) \leqq 2\pi\tilde{C}(\mathcal{O}_{n_j}) = c_{\mathcal{K}}(\mathcal{O}_{n_j}) \leqq c_{\mathcal{K}}(\mathcal{O}).$$

Letting $j \to \infty$ gives the reverse inequality and completes the proof of Proposition 14.

16. The projection π of the Royden compactification onto the Kuramochi compactification is the continuous mapping of $R^*_{\mathfrak{R}}$ onto $R^*_{\mathcal{K}}$ determined as follows: for each $p \in R^*_{\mathfrak{R}}$, $f(p) \in R^*_{\mathcal{K}}$ such that $f(\pi(p)) = f(p)$, for every $f \in N$. We shall use the following relationship between Dirichlet capacities on the two compactifications.

LEMMA. *For a compact set* $K \subset W^*_{\mathcal{K}}$

$$c_{\mathfrak{R}}(\pi^{-1}(K)) \leqq c_{\mathcal{K}}(K). \tag{23}$$

This is a simple consequence of the definition givven in (4) and the fact that $\pi \mid R$ is the identity. Indeed, let $\mathcal{O}$ be any open set in $W^*_{\mathcal{K}}$ with $\bar{\mathcal{O}} \subset W^*_{\mathcal{K}}$ and $K \subset \mathcal{O}$. Then $\pi^{-1}(\mathcal{O})$ is open in $W^*_{\mathfrak{R}}$ with $\overline{\pi^{-1}(\mathcal{O})} \subset W^*_{\mathfrak{R}}$ and $\pi^{-1}(K) \subset \pi^{-1}(\mathcal{O})$. Thus

$$c_{\mathfrak{R}}(\pi^{-1}(K)) \leqq d(\pi^{-1}(\mathcal{O}) \cap R) = d(\mathcal{O} \cap R).$$

Taking the infimum over all such $\mathcal{O}$ establishes (23).

Singular points in the Royden boundary

17. A point $p \in R^*$ is called *singular* if its Dirichlet capacity is positive. In view of Propositions 12 and 14 this definition is independent of the capacity used. As announced in the introduction we wish to characterize the existence of singular points in terms that do not depend on the compactification. We begin with the Royden compactification.

THEOREM. *Let $u_0 \in \mathrm{HBD}(W; \partial W)$ be nonzero. Then the linear functional given by* (A.1) *is multiplicative if and only if there is a singular point $p \in R^*_{\mathscr{R}}$ and u_0 is its Dirichlet capacitary potential.*

We begin by establishing a simple lemma. Set $M(\Delta) = M(R)|\Delta$. Note that $M(W; \partial W)|\Delta = M(\Delta)$, as well.

LEMMA. *If $\phi \in M(\Delta)$ and $\inf_\Delta \phi > 0$, then $1/\phi \in M(\Delta)$.*

For the proof choose $f \in M(R)$ with $f|\Delta = \phi$ and set $g = f \cup (\inf_\Delta \phi)$, which is in $M(R)$. Now $\inf_R g = \inf_\Delta \phi > 0$. Thus $1/g \in M(R)$ and $1/g|\Delta = 1/\phi$.

18. We proceed with the proof of the theorem. For each $\phi \in M(\Delta)$ we denote by f_ϕ a function in $M(W; \partial W)$ with $f_\phi|\Delta = \phi$. Let u_0 be a nonzero function in $\mathrm{HBD}(W; \partial W)$. We define

$$l(\phi) = \frac{1}{D(u_0)} D(f_\phi, u_0). \tag{24}$$

If f'_ϕ is another function with $f_\phi|\Delta = \phi$, then $f_\phi - f'_\phi \in M_\Delta(W; \partial W)$ and hence by (1) we see that $D(f_\phi, u_0) = D(f'_\phi, u_0)$; i.e. $l(\phi)$ is well defined.

We now assume that (A.1) is multiplicative. Then l is also a multiplicative linear functional. Since $l(u_0|\Delta) = 1$, it is nontrivial and, in particular, we must have $l(1) = 1$. Consider the maximal ideal $J = \{\phi \in M(\Delta); l(\phi) = 0\}$ of the algebra $M(\Delta)$. We claim that there exists a unique point $p \in \Delta$ such that

$$J = J_p, \tag{25}$$

where $J_p = \{\phi \in M(\Delta); \phi(p) = 0\}$. We first show that there is a point $p \in \Delta$ such that $\phi(p) = 0$ for every $\phi \in J$. If this were not the case, then for every $q \in \Delta$ there would exist a function $\phi_q \in J$ with $\phi_q(q) \neq 0$. Since any constant multiple of ϕ_q^2 also belongs to J, we may assume that $\phi_q \geqq 0$ on Δ and that $\phi_q(q) = 2$. Since

$$\bigcup_{q \in \Delta} \{r \in \Delta; \phi_q(r) > 1\} = \Delta,$$

there exist a finite number of points $q_1,\dots,q_m$ in Δ such that

$$\bigcup_{j=1}^{m} \{r \in \Delta; \phi_{q_j}(r) > 1\} = \Delta.$$

Then the function $\psi = \sum_{j=1}^{m} \phi_{q_j}$ belongs to J and $\psi > 1$ on Δ. Then by Lemma 17, $1/\psi \in M(\Delta)$ and hence $1 = (1/\psi)\psi \in J$, a contradiction. Thus there exists a point $p \in \Delta$ such that $J \subset J_p$. The maximality of J implies that $J = J_p$.

19. Since $\phi - \phi(p)$ belongs to J, we see that

$$0 = l(\phi - \phi(p)) = l(\phi) - \phi(p)l(1) = l(\phi) - \phi(p), \qquad \text{for every } \phi \in M(\Delta);$$

i.e., $l(\phi) = \phi(p)$ for every $\phi \in M(\Delta)$. Thus for every $f \in M(W;\partial W)$, $l(f \mid \Delta) = f(p)$; i.e.,

$$\frac{1}{D(u_0)} D(f, u_0) = f(p), \tag{26}$$

for every $f \in M(W;\partial W)$. We see from (26) that $1 = u_0(p)$. Take any $v \in M(W;\partial W)$ with $v(p) \geqq 1$. Then by (26) we see that $D(u_0) \leqq D(v, u_0)$. The Schwarz inequality, $D(v,u_0) \leqq D(u_0)^{1/2}D(v)^{1/2}$, implies that $D(u_0) \leqq D(v)$. We have shown that u_0 is the minimizing function in (17). Thus $c_{\mathscr{K}}(p) = \operatorname{cap}(p) > 0$ and u_0 is the Dirichlet capacitary potential for p.

Conversely, assume that u_0 is the Dirichlet capacitary potential of a point $p \in \Delta$. Then for an arbitrary $f \in M(W;\partial W)$ the function $f - f(p)u_0 \in M(W;\partial W)$ and vanishes at p. Thus by Lemma 13, $D(f - f(p)u_0, u_0) = 0$, which is (26). Since the functional $f \to f(p)$ is multiplicative, so is (A.1) and the proof is complete.

Singular points in the Kuramochi boundary

20. A function f is said to be *quasi-continuous* on $W^*_{\mathscr{K}}$ if for every $\varepsilon > 0$ there is an open subset $U_\varepsilon \subset W$ such that $c_{\mathscr{K}}(U_\varepsilon) < \varepsilon$ and $f \mid W^*_{\mathscr{K}} - U_\varepsilon$ is continuous. A function f is said to be *quasi-continuous* on $R^*_{\mathscr{K}}$ if $f \mid W^*_{\mathscr{K}}$ is quasi-continuous for every choice of $W^*_{\mathscr{K}}$. For a continuous function f on R we use f^* to denote a quasi-continuous function on $R^*_{\mathscr{K}}$ such that $f^* \mid R = f$ and call it a quasi-continuous extension of f. If $p \in R^*_{\mathscr{K}} - R$ has $c_{\mathscr{K}}(p) = 0$, then any value can be assigned to a quasi-continuous extension of f to R at p. This, however, is not the case when $c_{\mathscr{K}}(p) > 0$.

LEMMA. *Suppose $p \in R^*_{\mathscr{K}} - R$ has $c_{\mathscr{K}}(p) > 0$. If f is a continuous function on R and has a quasi-continuous extension f^* to $R^*_{\mathscr{K}}$, then the value $f^*(p)$ is uniquely* *determined for any choice of the quasi-continuous extension f^*. If g is another continuous function on R with a quasi-continuous extension g^* to $R^*_{\mathcal{K}}$ and λ is a real number, then λf, $f+g$, fg have quasi-continuous extensions to $R^*_{\mathcal{K}}$ satisfying $(\lambda f)^*(p)=\lambda f^*(p)$, $(f+g)^*(p)=\lambda f^*(p)$. $(fg)^*(p)=f^*(p)g^*(p)$.*

For the proof let $\varepsilon>0$ and $U_\varepsilon(f)$, $U_\varepsilon(g)$ be open sets in $R^*_{\mathcal{K}}$ with $c_{\mathcal{K}}(U_\varepsilon(f))<\varepsilon/2$, $c_{\mathcal{K}}(U_\varepsilon(g))<\varepsilon/2$ such that f^* and g^* are continuous on $R^*_{\mathcal{K}}-U_\varepsilon(f)$, $R^*_{\mathcal{K}}-U_\varepsilon(g)$, respectively. Let $U_\varepsilon=U_\varepsilon(f)\cup U_\varepsilon(g)$. Since $c_{\mathcal{K}}(U_\varepsilon)\leqq c_{\mathcal{K}}(U_\varepsilon(f))+c_{\mathcal{K}}(U_\varepsilon(g))$, we have $c_{\mathcal{K}}(U_\varepsilon)<\varepsilon$ and λf^*, f^*+g^*, f^*g^* are continuous on R^*-U_ε with $\lambda f^*\,|\,R=\lambda f$, $f^*+g^*\,|\,R=f+g$, $f^*g^*\,|\,R=fg$. Thus λf, $f+g$, fg have quasi-continuous extensions $(\lambda f)^*$, $(f+g)^*$, $(fg)^*$ to $R^*_{\mathcal{K}}$.

We now prove the uniqueness of the value of f^* at p. Choose $\varepsilon=c_{\mathcal{K}}(p)/2$ and let $\{V\}$ be the family of open sets in $R^*_{\mathcal{K}}$ containing p. Since $c_{\mathcal{K}}(V\cap R)=c_{\mathcal{K}}(V)\geqq c_{\mathcal{K}}(p)>\varepsilon$, $V\cap R$ cannot be contained in U_ε. Thus we can find a point $z_V\in(V\cap R)\cap(R^*_{\mathcal{K}}-U_\varepsilon)$. Since $\lim_V z_V=p$, by the continuity of f^* on $R^*_{\mathcal{K}}-U_\varepsilon$, we have

$$f^*(p)=\lim_V f^*(z_V)=\lim_V f(z_V),$$

and similarly

$$g^*(p)=\lim_V g^*(z_V)=\lim_V g(z_V).$$

If $f=g$ on R, then their extensions f^*, g^* satisfy $f^*(p)=g^*(p)$. If f and g are not necessarily the same, then the above shows that

$$(\lambda f)^*(p)=\lim_V \lambda f(z_V)=\lambda f^*(p),$$

$$(f+g)^*(p)=\lim_V (f+g)(z_V)=g^*(p)+f^*(p),$$

$$(fg)^*(p)=\lim_V (fg)(z_V)=g^*(p)f^*(p).$$

21. Every function in $M(R)$, and in particular in $M(W;\partial W)$, has a quasi-continuous extension to $R^*_{\mathcal{K}}$ (cf. [1, p. 191]). We shall need the following lemma concerning the relationship between this extension and its continuous extension to $R^*_{\mathcal{K}}$.

LEMMA. *Let $p\in R^*_{\mathfrak{R}}-R$ be singular. Then $q=\pi(p)\in R^*_{\mathcal{K}}-R$ is singular and $f^*(q)=f(p)$, for every $f\in M(W;\partial W)$.*

Note that by Lemma 16, $c_{\mathcal{K}}(q) \geqq c_{\mathcal{R}}(\pi^{-1}(q)) \geqq c_{\mathcal{R}}(p) > 0$, which means that q is singular. Set $\varepsilon = c_{\mathcal{R}}(p)/2$. Take $f \in M(W; \partial W)$ and U_ε open in W^* such that f^* is continuous on $R^*_{\mathcal{K}} - U_\varepsilon$ and $c_{\mathcal{K}}(U_\varepsilon) < \varepsilon$. Consider the family $\{V\}$ of open sets V in $R^*_{\mathcal{R}}$ with $p \in V$. In view of (9) and (10) we have $c_{\mathcal{K}}(V \cap R) = c_{\mathcal{R}}(V \cap R)$ and by Lemma 11, $c_{\mathcal{R}}(V \cap R) = c_{\mathcal{R}}(V) \geqq c_{\mathcal{R}}(p) = 2\varepsilon$. Thus we see that $V \cap R$ cannot be contained in U_ε. For each V we may choose a point $z_V \in (V \cap R) \cap (R^*_{\mathcal{K}} - U_\varepsilon)$. On the one hand, since $z_V \to p$ in $R^*_{\mathcal{R}}$, we have $f(z_V) \to f(p)$. On the other hand, since $z_V = \pi(z_V) \to \pi(p) = q$ in $R^*_{\mathcal{K}}$, and in fact in $R^*_{\mathcal{K}} - U_\varepsilon$, $f(z_V) = f^*(\pi(z_V)) \to f^*(q)$. Thus $f^*(q) = f(p)$.

22. We now establish a relationship between the Dirichlet capacitary potential and the Kuramochi capacitary potential.

PROPOSITION. *Let $p \in R^*_{\mathcal{R}} - R$ be singular and let u_0 be its Dirichlet capacitary potential. Then u_0 is the Kuramochi capacitary potential for $q = \pi(p) \in R^*_{\mathcal{K}} - R$.*

By Lemma 21, $u_0^*(q) = u_0(p) = 1$ and by Proposition 14, $\tilde{C}(q) > 0$. Let $\tilde{p}^\mu$ be the Kuramochi capacitary potential for q on $R^*_{\mathcal{K}}$. Then $\mu = (1/2\pi)D(\tilde{p}^\mu)\varepsilon_q$. By [1, Satz 17.3]

$$D(\tilde{p}^\mu, u_0) = 2\pi \int u_0^* d\mu = D(\tilde{p}^\mu). \tag{27}$$

Since $\tilde{p}^\mu \in M(W; \partial W)$, Lemma 21 gives, $\tilde{p}^\mu(q) = \tilde{p}^\mu(p)$ and consequently, $\tilde{p}^\mu(p) = 1$. Thus by (18) $D(u_0, u_0 - \tilde{p}^\mu) = 0$, i.e. $D(u_0) = D(u_0, \tilde{p}^\mu)$. By combining this with (27) we obtain

$$D(u_0 - \tilde{p}^\mu) = D(u_0) - D(\tilde{p}^\mu) = 0.$$

This implies that $u_0 = \tilde{p}^\mu$ and completes the proof.

23. With these preparations we are ready to prove the counterpart of Theorem 17 for the Kuramochi boundary.

THEOREM. *Let $u_0 \in \mathrm{HBD}(W; \partial W)$ be nonzero. Then the linear functional given by* (A.1) *is multiplicative if and only if there is a singular point $q \in R^*_{\mathcal{K}}$ and u_0 is its Kuramochi capacitary potential.*

We start by assuming that u_0 is the Kuramochi capacitary potential of a point $q \in R^*_{\mathcal{K}}$. Since u_0 is nonzero, we must have $\tilde{C}(q) > 0$. By the equilibrium principle there is a unique measure on the singleton $\{q\}$ such that its Kuramochi potential

$$\tilde{p}^\mu(z) = \int \tilde{g}(z, r) d\mu(r)$$

satisfies $\tilde{p}^{\mu} \leqq 1$ on W^*, $\tilde{p}^{\mu} = 1$ on $\{q\}$ and $\mu(q) = \tilde{C}(q)$. Then $u_0 = \tilde{p}^{\mu}$ and $\mu = c\varepsilon_q$, where ε_q is the point mass at q. Let $f \in M(W; \partial W)$ and f^* its quasi-continuous extension to $R^*_{\mathcal{K}}$. By (27) we have

$$D(f, u_0) = D(f, \tilde{p}^{\mu}) = 2\pi \int f^* \, d\mu = 2\pi c f^*(q).$$

Observe that $u_0(q) = \tilde{p}^{\mu}(q) = 1$. By setting $f = u_0$ in the above we obtain $D(u_0) = D(u_0, u_0) = 2\pi c u_0^*(q) = 2\pi c$. Hence

$$\frac{1}{D(u_0)} D(f, u_0) = f^*(q)$$

for every $f \in M(W; \partial W)$. By Lemma 20, $f \to f^*(q)$ is a multiplicative linear functional on $M(W; \partial W)$ and therefore (A.1) is multiplicative.

Conversely, we assume that (A.1) for u_0 is multiplicative. By Theorem 17 there exists a point $p \in R^*_{\mathcal{R}} - R$ with $c_{\mathcal{R}}(p) > 0$ such that u_0 is the Dirichlet capacitary potential for p. By Lemma 21 the point $q = \pi(p)$ has $c_{\mathcal{K}}(q) > 0$ and by Proposition 22 u_0 is its Kuramochi capacitary potential.

Connection between singular points

24. We have seen that the existence of a singular point on the Royden compactification implies that one also exists on the Kuramochi boundary. The following completely describes the behavior of singular points under the natural projection $\pi : R^*_{\mathcal{R}} \to R^*_{\mathcal{K}}$.

THEOREM. *A point $q \in R^*_{\mathcal{K}} - R$ is singular if and only the fiber $\pi^{-1}(q)$ contains a singular point. For any $q \in R^*_{\mathcal{K}} - R$, the fiber $\pi^{-1}(q)$ contains at most one singular point.*

As we noted above the necessity was established in Lemma 21. For the sufficiency assume that q is singular and let $u_0 = \tilde{p}^{\mu}$ be its Dirichlet capacitary potential. Then according to Theorem 23 (A.1) is multiplicative and consequently by Theorem 17 there is a point $a \in R^*_{\mathcal{R}} - R$ such that u_0 is its Dirichlet capacitary potential. Set $\pi(a) = r$. Then by Proposition 22 u_0 is the capacitary potential for r. Hence $r = q$.

For the proof of the last assertion of the theorem assume that there are two points a and b in $\pi^{-1}(q)$ which are singular. They by Proposition 22 the Dirichlet capacitary potentials u_a and u_b for a and b are equal to the capacitary potential u_0 for q on R. Let $f \in M(W; \partial W)$ with $f(a) = 0$ and $f(b) = 1$. Since $D(f, u_a) = D(u_a) f(a) = 0$ and $D(f, u_b) = D(u_b) f(b) = D(u_b)$, we have a contradiction to the fact that $D(f, u_a) = D(f, u_0) = D(f, u_b)$.

References

1. C. Constantinescu und A. Cornea, *Ideale Ränder Riemannscher Flächen*, Springer-Verlag, 1963.

2. M. Glasner and M. Nakai, *Singular points on the Royden harmonic boundary*, Complex Variables, to appear.

3. Z. Kuramochi, *Examples of singular points*, J. Fac. Sci. Hokkaido Univ., Ser. 1, **16** (1962), 149–187.

4. Z. Kuramochi, *On Iversen's property and the existence of bounded analytic functions*, Hokkaido Math. J. **12** (1983), 147–198.

5. L. Sario and M. Nakai, *Classification Theory of Riemann Surfaces*, Springer-Verlag, 1970.